ANNUITIES AND INDIVIDUAL WELFARE

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Abstract

This paper advances the theory of annuity demand. First, we derive sufficient conditions under which complete annuitization is optimal, showing that this well-known result holds true in a more general setting than in Yaari (1965). Specifically, when markets are complete, sufficient conditions need not impose exponential discounting, intertemporal separability or the expected utility axioms; nor need annuities be actuarially fair, nor longevity risk be the only source of consumption uncertainty. All that is required is that consumers have no bequest motive and that annuities pay a rate of return for survivors greater than those of otherwise matching conventional assets, net of administrative costs. Second, we show that full annuitization may not be optimal when markets are incomplete. Some annuitization is optimal as long as conventional asset markets are complete. The incompleteness of markets can lead to zero annuitization but the conditions on both annuity and bond markets are stringent. Third, we extend the simulation literature that calculates the utility gains from annuitization by considering consumers whose utility depends both on present consumption and a "standard-of-living" to which they have become accustomed. The value of annuitization hinges critically on the size of the initial standard-of-living relative to wealth.

Key Words: Annuities, annuitization, Social Security, pensions, longevity risk, insurance, standard-of-living, habit.

1 Introduction

Providing a secure source of retirement income is an issue of increasing importance to individuals and policy-makers alike. The most common retirement age for a male in the United States today is 62 years [1] and, thanks to the substantial reduction in mortality risk at older ages witnessed over the past century, expected remaining life span for a 62 year old male is nearly 19 years – almost to age 81.[2] There is, however, substantial uncertainty around this expected value. Approximately 16 percent of 62 year old males will die before age 70, while another 16 percent will live to age 90 or beyond. As a result, longevity risk - uncertainty about how long one will live - is a substantial source of financial uncertainty facing today's retirees. Consideration of couples extends the upper tail of life expectancy outcomes.

Since the seminal contribution of Yaari (1965) on the theory of a life-cycle consumer with an unknown date of death, annuities have played a central role in economic theory. His widely cited result is that certain consumers should annuitize all of their savings. However, these consumers were assumed to satisfy several very restrictive assumptions: they were von Neumann-Morgenstern expected utility maximizers with intertemporally separable utility, they faced no uncertainty other than time of death and they had no bequest motive. In addition, the annuities available for purchase by these individuals were assumed to be actuarially fair. While the subsequent literature on annuities has occasionally relaxed one or two of these assumptions, the "industry standard" is to maintain most of these conditions. In particular, the literature has universally retained expected utility and additive separability, the latter dubbed "not a very happy assumption" by Yaari.

This paper advances the theory of annuity demand in several directions. Section 2 derives sufficient conditions for complete annuitization to be optimal, demonstrating that this well-known result holds true in a much more general setting than that in Yaari (1965). Specifically, we show that when markets are complete, it is not necessary for consumers to be exponential discounters, for utility to obey expected utility axioms or be intertemporally separable, or for annuities to be actuarially fair. Rather, all that is required for complete annuitization to be optimal is that consumers have no bequest motive and that annuities pay a rate of return to survivors, net of administrative costs, that is greater than the return on conventional assets of matching financial risk. Section 2.1 considers a two period setting with no uncertainty other than date of death, in which all trade occurs at once. Here, all savings are annuitized so long as there is no bequest and annuities have a higher return for survivors than conventional

[1] Gustman and Steinmeier (2002)

[2] *2002 Annual Report of the Board of Trustees of the Federal Old-Age and Survivors Insurance and Disability Insurance Trust Funds* (2002)

assets. Section 2.2 extends this result to the Arrow-Debreu case with arbitrarily many future periods with aggregate uncertainty, as long as conventional asset and annuity markets are complete.

Despite this strong theoretical prediction, few people voluntarily annuitize outside of Social Security and defined benefit plans. To provide theoretical guidance on why this so called "annuity puzzle" might exist, in section 3 we show how the full annuitization result can break down when markets for either annuities or conventional assets are incomplete. Section 3.1 examines the case where conventional markets are complete but annuity markets are incomplete. We derive the weaker result that as long as trade occurs all at once and preferences are such that consumers avoid zero consumption in every state of nature, then consumers will always annuitize at least part of their wealth. Also, if trade occurs all at once, we derive the result that an annuitized version of any conventional asset will always dominate the underlying asset for consumers with no bequest motive, even if the asset does not pay off in every state of nature. An important consequence of this result is that the finding that "annuities dominate conventional assets" extends past riskless bonds to risky securities such as stocks or mutual funds.

A practical implication of these results is that "variable life annuities" may dominate mutual funds, provided that the higher expenses associated with variable annuities are not too high.[3] For example, suppose the provider of a mutual fund family doubles the number of available funds by offering a matching annuitized fund that periodically takes the accounts of investors who die and distributes the proceeds across the accounts of surviving investors.[4] The returns to this annuitized fund will strictly exceed the returns of the underlying fund for surviving investors.

Section 3.2 considers situations in which it can be optimal not to annuitize any wealth at all. A key finding of this section is that under plausible conditions on returns,[5] incompleteness of conventional asset markets as well as incompleteness of annuity markets themselves, is required for zero annuitization to be optimal. This highlights the common observation that part of the solution to the annuity puzzle may lie in the lack of complete insurance against other types of risk. Section 3.2.3 sharpens this observation by showing an example where a critical role is played by a decrease in the possible maximal date of death.

[3]We refer to true life annuities, not the variable annuities widely marketed that contain only an annuitization option.

[4]TIAA-CREF currently provides annuities with such a structure.

[5]Milevsky and Young (2002) considers a violation of the return condition that may render zero annuitization optimal. In particular, in the absence of variable annuities they demonstrate that it can be optimal to defer annuitization, with deferral more attractive as risk tolerance grows.

Section 4 extends the simulation literature,[6] that calculates the utility gains from annuitization by considering consumers whose utility depends both on present consumption and a "standard-of-living" to which they have become accustomed.[7] In our specification, whether annuities are more or less valuable under this standard-of-living model than under the conventional model hinges on whether the initial standard-of-living is large relative to retirement resources.[8] In particular, if the initial standard-of-living at the start of retirement is large relative to the individuals stock of resources, complete annuitization in the form of a constant real annuity is not optimal, since it does not allow the individual to optimally "phase down" from the pre-retirement level of consumption to which she had become accustomed. If, however, the stock of retirement wealth is large relative to the standard-of-living, annuities are even more valuable than in the usual model of separability.

Section 5 concludes and proposes directions for future research.

2 When is Complete Annuitization Optimal?

The literature on annuities has long been concerned with the "annuity puzzle." This puzzle consists of the combination of Yaari's finding that, under certain assumptions, complete annuitization is optimal with the fact that outside of Social Security and defined benefit pension plans, very few U.S. consumers voluntarily annuitize any of their private savings.[9] This issue is of interest from a theoretical perspective because it bears upon the issue of how to model consumer behavior in the presence of uncertainty. It is also of policy interest because of the gradual shift in the US from defined benefit plans, which typically pay out as an an annuity, to defined contribution plans, that often do not require, or even offer, retirees the opportunity to annuitize. The role of annuitization is also important in national defined contribution plans, which have been growing in importance. This section of the paper adds to the annuity puzzle by deriving much more general conditions under which full annuitization is optimal. Section 3 will then shed light on potential resolutions to the puzzle

[6]See for example Kotlikoff and Spivak (1981), Friedman and Warshawsky (1990), Yagi and Nishigaki (1993) and Mitchell, Poterba, Warshawsky and Brown (1999)

[7]We use the formulation in Diamond and Mirrlees (2000). This formulation involves what is sometimes referred to as an "internal habit." Different models of intertemporal dependence in utility are discussed in, for example, Dusenberry (1949), Abel (1990), Constantinides (1990), Deaton (1991), Campbell and Cochrane (1999), Campbell (2002) and Gomes and Michaelides (2003).

[8]This might occur due to myopic failure to save, or due to adverse health or financial shocks.

[9]This assertion is consistent with the large market for what are called variable annuities since these insurance products do not include a commitment to annuitize accumulations, nor does there appear to be much voluntary annuitization. See for example Brown and Warshawsky (2001).

by examining market incompleteness.

2.1 Annuity Demand in a Two Period Model with No Aggregate Uncertainty

Analysis of intertemporal choice is greatly simplified if resource allocation decisions are made all at once. Consumers will be willing to commit to a fixed plan of expenditures at the start of time under either of two conditions. The first condition, standard in the complete market Arrow-Debreu model is that, at the start of time, consumers are able to trade goods across time and all states of nature. Alternatively, first period asset trade obviates future trade across states of nature if consumers live for only two periods.

Yaari considered annuitization in a continuous time setting where consumers are uncertain only about the date of death. Some results, however, can be seen more simply by dividing time into two discrete periods: the present, period 1, when the consumer is definitely alive and period 2, when the consumer is alive with probability $1-q$. We maintain the assumption that there is no bequest motive and for the moment assume that only survival to period 2 is uncertain. In this case, lifetime utility is defined over first period consumption c_1 and planned consumption in the event that the consumer is alive in period 2, c_2. By writing

$$U = U(c_1, c_2)$$

we allow for the possibility that the effect of second-period consumption on utility depends on the level of first period consumption. This formulation does not require that preferences satisfy the axioms for U to be an expected value.

We approach both optimal decisions and the welfare evaluation of the availability of annuities by taking a dual approach. That is, we analyze consumer choice in terms of minimizing expenditures subject to attaining at least a given level of utility. We measure expenditures in units of first period consumption. Assume that there is a bond available which returns R_B units of consumption in period 2, whether the consumer is alive or not, in exchange for each unit of the consumption good in period 1. Assume, in addition, the availability of an annuity which returns R_A in period 2 if the consumer is alive and nothing if the consumer is not alive. Whereas the bond requires the supplier to pay R_B whether or not the saver is alive,[10] the annuity pays out only if the saver is alive. If the annuity were actuarially fair, then we would have $R_A = \frac{R_B}{1-q}$. Adverse selection and higher transaction costs for paying annuities than for paying bonds may drive returns below this level. However, because any consumer will

[10]These values should be interpreted as net of the transaction cost of a consumer buying these assets.

have a positive probability of dying between now and any future period, thereby relieving borrowers' obligation, we regard the following as a weak assumption:[11]

Assumption 1 $R_A > R_B$

Denoting by A savings in the form of annuities and by B savings in the form of bonds, if there is no other income in period 2 (e.g. retirees), then

$$c_2 = R_A A + R_B B, \tag{1}$$

and expenditures for lifetime consumption are

$$E = c_1 + A + B. \tag{2}$$

The expenditure minimization problem can thus be defined as a choice over first period consumption and bond and annuity holdings:

$$\min_{c_1, A, B} c_1 + A + B \tag{3}$$

$$s.t. \, U(c_1, R_A A + R_B B) \geq \bar{U}$$

By Assumption 1, purchasing annuities and selling bonds in equal numbers would cost nothing and yield positive consumption when alive in period 2 but leave a debt if dead. However, such an arbitrage would imply that lenders would be faced with losses in the event that such a trader failed to live to period 2. The standard Arrow-Debreu assumption is that planned consumption is in the consumption possibility space. For someone who is dead, this would require that the consumer not be in debt. In this simple setting the restriction is therefore that

$$B \geq 0.$$

This setup yields two important results. The first considers improving an arbitrary allocation while the second refers to the optimal plan.

Result 1 *(i) If $B > 0$, then (i) annuitization can be increased while reducing expenditures and holding the consumption vector constant. (ii) The solution to problem (3) sets $B = 0$.*

Proof. *For (i) a sale of $\frac{R_A}{R_B}$ of the bond and purchase of 1 annuity works by Assumption 1 and the definition of c_2. For (ii), by(i), a solution with $B > 0$ fails to minimize expenditures. Solutions with the inequality reversed are not permitted.* ■

[11] That $R_B < R_A < \frac{R_B}{1-q}$ is supported empirically by Mitchell et al. (1999). If the first inequality were violated, annuities would be dominated by bonds.

In this two period setting, Part (ii) of Result 1 is an extension of Yaari's result of complete annuitization to conditions of intertemporal dependence in utility, preferences that may not satisfy expected utility axioms and actuarially unfair annuities. All that is required is that there is no bequest motive and that the payout of annuities dominates that of conventional assets for a survivor.

Part (i) of Result 1 implies that the introduction of annuities reduces expenditures for constant utility, thereby generating increased welfare (a positive equivalent variation or a negative compensating variation). We might be interested in two related calculations: the reduction in expenditures associated with allowing consumers to annuitize a larger fraction of their savings (particularly from a level of zero) and the benefit associated with allowing consumers to annuitize all of their savings. That is, we want to know the effect on the expenditure minimization problem of loosening or removing an additional constraint on problem (3) that limits annuity opportunities. To examine this issue, we restate the expenditure minimization problem with a constraint on the availability of annuities as:

$$\min_{c_1,A,B} : c_1 + A + B \tag{4}$$

$$s.t. : U(c_1, R_A A + R_B B) \geq \bar{U}$$

$$A \leq \bar{A}. \tag{5}$$

$$B \geq 0 \tag{6}$$

We know that utility maximizing consumers will take advantage of an opportunity to annuitize as long as second-period consumption is positive. Positive planned consumption is ensured by the plausible condition that zero consumption is extremely bad:

Assumption 2

$$\lim_{c_t \to 0} \frac{\partial U}{\partial c_t} = \infty \quad for \ \ t = 1, 2$$

We can see from the optimization (4) that allowing consumers previously unable to annuitize any wealth to place a small amount of their savings into annuities (incrementing $\bar{A}$ from zero) leaves second period consumption unchanged (since the cost of the marginal second-period consumption is unchanged and so too, therefore, is the optimal level of consumption in both periods). By Result 1, in this case, a small increase in $\bar{A}$ generates a very small substitution of the annuity for the bond proportional to the prices

$$\frac{dA}{d\bar{A}} = 1$$

$$\frac{dB}{d\bar{A}} = -\frac{R_A}{R_B},$$

leaving consumption unchanged: $dc_2 = R_A dA + R_B dB = R_A - R_A \frac{R_B}{R_B} = 0$.

The effect on expenditures is equal to $(1 - \frac{R_A}{R_B}) < 0$. This is the welfare gain from increasing the limit on available annuities for an optimizing consumer with positive bond holdings.

If constraint (5) is removed altogether, the price of second period consumption in units of first period consumption falls from $\frac{1}{R_B}$ to $\frac{1}{R_A}$. With a change in the cost of marginal second-period consumption, its level will adjust. Thus the cost savings is made up of two parts. One part is the savings while financing the same consumption bundle as when there is no annuitization and the second is the savings from adapting the consumption bundle to the change in prices. We can measure the welfare gain in going from no annuities to potentially unlimited annuities by integrating the derivative of the expenditure function between the two prices:

$$E|_{\bar{A}=0} - E|_{\bar{A}=\infty} = -\int_{R_A^{-1}}^{R_B^{-1}} c_2(p_2) dp_2, \tag{7}$$

where c_2 is compensated demand arising from minimization of expenditures equal to $c_1 + c_2 p_2$ subject to the utility constraint without a distinction between asset types.

Equation (7) implies that consumers who save more (have larger second-period consumption) benefit more from the ability to annuitize completely:

Result 2 *The benefit of allowing complete annuitization (rather than no annuitization) is greater for consumer i than for consumer j if consumer i's compensated demand for second period consumption (equivalently, compensated savings) exceeds consumer j's for any price of second period consumption.*

2.2 Extending the Model to Many Periods and States with Complete Markets

While a two-period model with no aggregate uncertainty offers the virtue of simplicity, real consumers face a more complicated decision setting. In particular, they face many periods of potential consumption and each period may have several possible states of nature. For example, a 65 year old consumer has some probability of surviving to be a healthy and active 80 year old, some chance of finding herself sick and in a nursing home at age 80 and some chance of not being alive at all at age 80. Moreover, rates of return on some assets are stochastic.

The result of the optimality of complete annuitization survives subdivision of the aggregated future defined by c_2 into many future periods and states. A particularly simple subdivision would be to add a third period, so that survival to period 2 occurs with probability $1 - q_2$ and to period 3 with probability $(1 - q_2)(1 - q_3)$. In this case, bonds and annuities which pay out separately in period 2 with rates R_{B2} and R_{A2}, and period 3 with rates R_{B3} and R_{A3} are sufficient to obviate trade in periods 2 or 3. That is, defining bonds and annuities purchased in period 1 with the appropriate subscript,[12]

$$E = c_1 + A_2 + A_3 + B_2 + B_3$$

$$c_2 = R_{B2}B_2 + R_{A2}A_2,$$

$$c_3 = R_{B3}B_3 + R_{A3}A_3.$$

If Assumption 1 is modified to hold period by period, Result 1 extends trivially. Note that we have set up what we will call "Arrow bonds" (here B_2 and B_3) by combining two states of nature that differ in no other way except whether this consumer is alive. "Arrow annuities" which also recognize whether this consumer is alive complete the set of true Arrow securities of standard theory.

In order to take the next logical step, we can continue to treat c_1 as a scalar and interpret c_2, B_2, and A_2 as vectors with entries corresponding to arbitrarily many (possibly infinity) future periods ($t \leq T$), within arbitrarily many states of nature ($\omega \leq \Omega$). R_{A2} (R_{B2}) is then a matrix with columns corresponding to annuities (bonds) and rows corresponding to payouts by period and state of nature. Thus, the assumption of no aggregate uncertainty can be dropped. Multiple states of nature might refer to uncertainty about aggregate issues such as output, or individual specific issues beyond mortality such as health.[13] In order to extend the analysis, we need to assume that the consumer is sufficiently "small" that for each state of nature where the consumer is alive, there exists a state where the consumer is

[12]Implicitly, we are assuming that if markets reopened, the relative prices would be the same as are available in the initial trading period

[13]For a discussion of annuity payments that are partially dependent on health status, see Warshawsky, Spillman and Murtaugh (forthcoming).

dead and the equilibrium prices are otherwise identical. Completeness of markets still allows construction of Arrow bonds which represent the combination of two Arrow securities.

Annuities with payoffs in only one event state are contrary to our conventional perception of (and name for) annuities as paying out in every year until death. However, with complete markets, separate annuities with payouts in each year can be combined to create such securities. It is clear that the analysis of the two-period model extends to this setting, provided we maintain the standard Arrow-Debreu market structure and assumptions that do not allow an individual to die in debt. In addition to the description of the optimum, the formula for the gain from allowing more annuitization holds for state-by-state increases in the level of allowed level of annuitization. Moreover, by choosing any particular price path from the prices inherent in bonds to the prices inherent in annuities, we can measure the gain in going from no annuitization to full annuitization. This parallels the evaluation of the price changes brought about by a lumpy investment (see Diamond and McFadden (1974)).

In this section, we have extended the Yaari result of complete annuitization to conditions of aggregate uncertainty, actuarially unfair (but positive) annuity premiums and intertemporally dependent utility that need not satisfy the expected utility axioms. We have also shown that increasing the extent of available annuitization increases welfare for individuals who hold conventional bonds.[14] These results deepen the annuity puzzle by demonstrating that complete annuitization is optimal under a wider range of assumptions about individual preferences. Thus, given available empirical evidence about the small size of the private annuity market, a natural question is: when might individuals not fully annuitize? This is explored in the next section.

3 When Is Partial Annuitization Optimal?

In Section 2, we explored annuity demand in a setting with complete Arrow securities - both Arrow bonds and Arrow annuities were assumed to exist for every event. With such complete markets and without a bequest motive, the sufficient conditions for full optimization were very weak - just that the added costs of administering annuities were less than the value of security payments not made because of the deaths of investors. The full annuitization result depends on market completeness. In settings without market completeness, we

[14]The generalization of Result 2 to this case requires the very strong condition that after the present, consumption for agent i exceeds that of agent j state of nature by state of nature. That i's consumption grows at a greater rate than j's is not sufficient: allowing complete annuitization may yield reduction in many different prices by increasing any of many ratios $\frac{A_{t\omega}}{A_{t\omega}+B_{t\omega}}$. In general, these price changes are non-monotonic in time past period 1.

consider sufficient conditions for partial annuitization - the inclusion of some annuitization in optimized demand. We consider two alternative tpes of annuity market incompleteness. First, we consider a setting with complete Arrow bonds but only some Arrow annuities. Then we consider a setting with complete Arrow bonds and compound annuities - ones that involve payoffs in many events rather than being Arrow annuities. The first setting relates the annuity puzzle to the circumstance that insurance firms provide limited opportunities for annuitization. The second setting explores the puzzle in annuity demand given the annuity products that do exist.

3.1 Incomplete Annuity Markets (When Trade Occurs Once)

3.1.1 Incomplete Arrow Annuities

The logic of the argument in Section 2 was straightforward. Whenever there was a purchase of an Arrow bond, the cost of meeting a given utility level could be reduced by substituting purchase of an Arrow annuity for an Arrow bond. Thus with complete sets of both Arrow bonds and Arrow annuities, no Arrow bond would be purchased, implying that all of savings was invested in Arrow annuities. This line of argument will not result in complete annuitization if the set of Arrow annuities is not complete. That is, if the only way to get consumption in some future event is by purchasing an Arrow bond (since no Arrow annuity exists for that event), then some purchase of Arrow bonds for that event will be part of the optimum when the optimum has positive consumption in that event. Conversely, as long as any Arrow annuities exist, the optimum will include some annuitization.

3.1.2 Incomplete Compound Annuities

Most real world annuity markets require that a consumer purchase a particular time path of payouts, thereby combining in a single security a particular "compound" combination of Arrow securities. For example, the U.S. Social Security system provides annuities that are indexed to the Consumer Price Index and thus offer a constant real payout (ignoring the role of the earnings test). Privately purchased immediate life annuities are usually fixed in nominal terms, or offer a predetermined nominal slope such as a 5 percent nominal increase per year. Variable annuities link the payout to the performance of a particular underlying portfolio of assets and combine Arrow securities in that way. CREF annuities are also participating, which means that the payout also varies with the actual mortality experience for the class of investors.

Numerous simulation studies have examined the utility gains from annuities with these types of payouts that combine Arrow securities in a particular way. To consider such lifetime annuities in this setup, we continue to assume a double set of states of nature, differing only in whether the particular consumer we are analyzing is alive. We continue to assume a complete set of Arrow bonds and consider the effect of the availability of particular types of annuities. We also need to consider whether the return from annuities and bonds can be reinvested (markets are open) or must be consumed (markets are closed) In general, we will lose the result that complete annuitization is optimal. Nevertheless, we will get optimality of complete annuitization of initial savings in real annuities satisfying the return condition provided that optimal consumption is rising over time and markets for bonds are open. In a more general setting we examine sufficient conditions for the result that the optimal holding of annuities is not zero.

To illustrate these points, we consider a three-period model with no aggregate uncertainty and a complete set of bonds. Then we will show how the results generalize. If there are no annuities, then the expenditure minimization problem is:

$$\begin{gathered}\min_{c_1,A,B} : c_1 + B_2 + B_3 \\ s.t. : U(c_1, R_{B2}B_2, R_{B3}B_3) \geq \bar{U}\end{gathered} \tag{8}$$

That is, we have:

$$c_2 = R_{B2}B_2,$$

$$c_3 = R_{B3}B_3.$$

With the assumption of infinite marginal utility at zero consumption, all three of c_1, B_2 and B_3 are positive. Now assume that there is a single available annuity, A, that pays given amounts in the two periods. Assume further that there is no opportunity for trade after the initial contracting. The minimization problem is now

$$\begin{gathered}\min_{c_1,A,B} : c_1 + B_2 + B_3 + A \\ s.t. : U(c_1, R_{B2}B_2 + R_{A2}A, R_{B3}B_3 + R_{A3}A) \geq \bar{U} \\ c_2 = R_{B2}B_2 + R_{A2}A, \\ c_3 = R_{B3}B_3 + R_{A3}A.\end{gathered} \tag{9}$$

Before proceeding, we must revise the superior return condition for Arrow annuities that $R_{At\omega} > R_{Bt\omega} : \forall t\omega$. A more appropriate formulation for the return on a complex security that combines Arrow securities to exceed bond returns is that for any quantity of the payout stream provided by the annuity, the cost is less if bought with the annuity than if the same stream is bought through bonds. Define by ℓ a row vector of ones with length equal to the number of states of nature distinguished by bonds, let the set of bonds continue to be represented by a vector with elements corresponding to the columns of the matrix of returns R_B and let R_A be a vector of annuity payouts multiplying the scalar A to define state-by-state payouts.

Assumption 3 *For any annuitized asset A and any collection of conventional assets B, $R_A A = R_B B \Rightarrow A < \ell B$.*

For example, if there is an annuity that pays R_{A2} per unit of annuity in the second period and R_{A3} per unit of annuity in the third period, then we would have $1 < \left(\frac{R_{A2}}{R_{B2}} + \frac{R_{A3}}{R_{B3}}\right)$. By linearity of expenditures, this implies that any consumption vector that may be purchased strictly through annuities is less expensive when financed strictly through annuities than when purchased by a set of bonds with matching payoffs.[15]

Given the return assumption and the presence of positive consumption in all periods, it is clear that the cost goes down from the introduction of the first small amount of annuity, which can always be done without changing consumption. Thus we can also conclude that the optimum (including the constraint of not dying in debt) always includes some annuity purchase. It is also clear that full annuitization may not be optimal if the implied consumption pattern with complete annuitization is worth changing by purchasing a bond. That is, denoting partial derivatives of the utility function with subscripts, optimizing first period consumption given full annuitization, we would have the first order condition:

$$U_1(c_1, R_{A2}A, R_{A3}A) = R_{A2}U_2(c_1, R_{A2}A, R_{A3}A) + R_{A3}U_3(c_1, R_{A2}A, R_{A3}A).$$

Purchasing a bond would be worthwhile if we satisfy either of the conditions:

$$U_1(c_1, R_{A2}A, R_{A3}A) < R_{B2}U_2(c_1, R_{A2}A, R_{A3}A) \tag{10}$$

or

$$U_1(c_1, R_{A2}A, R_{A3}A) < R_{B3}U_3(c_1, R_{A2}A, R_{A3}A) \tag{11}$$

By our return assumption, we can not satisfy both of these conditions at the same time, but we might satisfy one of them. That is, the optimum will involve holding some of the annuitized asset and may involve some bonds, but not all of them.

[15]This assumption leaves open the possibility considered below that both bond and annuity markets are incomplete and some consumption plans can be financed only through annuities.

It is clear that these results generalize to a setting with complete Arrow bonds and some compound annuities with many periods and many states of nature. We show below that expenditure minimization requires that there must be positive purchases of at least one annuity.

Lemma 1 *Consider an asset $A*$ with finite, non-negative payouts R_{A*}. Any consumption plan $[c_1\, c_2]'$ with positive consumption in every state of nature can be financed by a combination of first period consumption, a positive holding of $A*$ and another strictly non-negative consumption plan.*

Proof. *Define $\bar{R}_{A*} = [\frac{1}{R_{A*21}}, ..., \frac{1}{R_{A*t\omega}}, ... \frac{1}{R_{A*T\Omega}}]'$ and define the scalar $\alpha = \min(c_2 \cdot \bar{R}_{A*})$. Now $c_2 = R_{A*}\alpha + Z$, where Z is weakly positive.* ■

We now have a weaker version of Result 1:

Result 3 *If marginal utilities are infinite at zero consumption (Assumption 2 holds) and there exist annuities with non-negative payouts which satisfy Assumption 3, then (i) when no annuities are held, a small increase in annuitization reduces expenditures, holding utility constant. Also, then (ii) expenditure minimization implies positive holdings of at least one annuity.*

Proof. *Suppose that the optimal plan (c_1, A, B) features $A = 0$. Then there are two possibilities: first, consumption might be zero in some future state of nature. By Assumption 2 this implies infinitely negative utility and fails to satisfy the utility constraint. If consumption is positive in every state of nature, then consumption is a linear combination of all strictly positive linear combinations of the Arrow bonds. But then since some strictly positive consumption plan can be financed by annuities, by Assumption 3 and Lemma 1, expenditures can be reduced holding consumption constant by a trade of some linear combination of the bonds for some combination of annuities with strictly positive payouts. This contradicts optimality of the proposed solution.* ■

Part (i) of Result 3 states that if consumers are willing to commit to lifetime expenditures all at once, then starting from a position of zero annuitization, a small purchase of *any* annuity (with a good return) increases welfare. This applies to any annuity with returns in excess of the underlying nonannuitized assets, no matter how distasteful the payout stream. Part (ii) is the corollary that optimal annuity holding is always positive. Lemma 1 shows that up to some point, annuity purchases do not distort consumption, so that their only effect is to reduce expenditures, as in the case where annuities markets are complete. When a large fraction of savings is annuitized, if the supply of annuitized assets fails to match demand, annuitization distorts consumption and some conventional assets may be preferred. From the proof of Result 3, it follows that the annuitized version of any conventional asset (with higher returns) that might be part of an optimal portfolio dominates the underlying asset.

3.2 Incomplete Annuity Markets With Trade More than Once

The setup so far has not allowed a second period of trade. However, if the existing annuities' payout trajectories are unattractive, households may wish to modify the consumption plan yielded by the dividend flows purchased at retirement through trade at later dates. We find in this case that positive annuitization remains optimal as long as conventional markets are complete and a revised form of the superior returns to annuitization condition holds. With incomplete conventional markets, it is possible for liquidity concerns to render zero annuitization optimal.

3.2.1 Trade in Many Periods with Complete Conventional Markets

Suppose that trade in bonds is allowed after the first period, with bond prices consistent with the returns that were present for trade before the first period. To begin, we assume that there is not an annuity available at any future trading time and that the consumer can save out of annuity receipts but can not sell the remaining portion of the annuity. Since there would be no further trade without an annuity purchase out of initial wealth, the optimum without any annuity is unchanged. Utility at the optimum, assuming some annuity purchase and consumption of the annuity return, raises welfare as above. Thus we conclude that the result that some annuity purchase is optimal (Result 3) carries over to the setting with complete bond markets at the start and further trading opportunities in bonds that involve no change in the terms of bond transactions. The possibility of reinvesting annuity returns can further enhance the value of annuity purchases and may result in the optimality of full annuitization.

Returning to the three period example with no uncertainty beyond individual mortality, a sufficient condition for complete annuitization at the start is that the consumption stream associated with complete annuitization at the first trading point was such that the individual would wish to save, rather than dissave. This is true even if one of the inequalities (10) or (11) is violated. To examine this issue, we now set up the expenditure minimization problem with retrading, denoting saving at the end of the second period by Z.

$$\min_{c_1,A,B,Z} : c_1 + B_2 + B_3 + A \tag{12}$$
$$s.t. : U(c_1, R_{B2}B_2 + R_{A2}A - Z, R_{B3}B_3 + R_{A3}A + (R_{B3}/R_{B2})\, Z) \geq \bar{U}.$$

The restriction of not dying in debt is the nonnegativity of consumption if A is set equal to zero: [16]

$$B_2,\ B_3,\ Z \geq 0$$
$$\begin{aligned} R_{B2}B_2 &\geq 0 \\ R_{B3}B_3 + (R_{B3}/R_{B2})\, Z &\geq 0 \end{aligned}$$

The assumption that dissaving would not be attractive given full annuitization is

$$R_{B2}U_2(c_1, R_{A2}A, R_{A3}A) \leq R_{B3}U_3(c_1, R_{A2}A, R_{A3}A) \tag{13}$$

This condition can be readily satisfied for preferences satisfying a suitable relationship between (implicit) utility discount rates and interest rates and the result extends with many future periods, as long as trade is allowed in each. To show this, we consider as a special case a world with $T-1$ future periods and no uncertainty except individual mortality, so that future consumption conditional on survival can be described by a vector c_2 with one element for each period up to T, beyond which no individual survives: $c_2 = [c_2, c_3 ... c_T]'$. Consumers have access to "Arrow" bonds and a single annuity product which pays out a constant real amount of $R_A A$ per period, where A is the amount of the annuity purchased in period 1. We assume that no annuities are available after the first period, but that future bond trades are allowed. By completeness of bond markets, we can consider the set of bonds to be described by $T-1$ securities, each of which pays out at a rate of $(1+r)^{t-1}$ at date t only. We assume further that there is a constant real interest rate of r on bonds and that the rate of return condition (Assumption 3) is satisfied. That is, the internal rate of return on the annuity, with periodic payouts multiplied by survival probabilities, exceeds r.

Because Assumptions 2 (infinite disutility from zero consumption in any future period) and 3 (any consumption plan that can be financed by annuities alone is financed most cheaply by annuities alone) are met:

[16] B_3 can be negative if Z is positive. However, a budget-neutral reduction in Z and increase in B_3, holding A constant, then yields equivalent consumption, so there is no restriction in disallowing negative B_3. If B_3 is non-negative, then Z must be zero as long as B_2 is positive, or else constant consumption with reduced expenditures could be obtained at a lower price by reducing B_2 and increasing A. That is, there are no savings out of bonds.

Result 4 *The solution to the expenditure minimization problem with markets as described above features $A > 0$.*

Proof. *Follows immediately from Result 3.* ■

By the no bankruptcy constraint, consumers may undo annuitization by saving if annuitization renders consumption too weighted towards early periods, but not by borrowing if annuitization renders consumption too weighted to later periods. With bonds liquid, the liquidity constraint given a constant real annuity requires that expenditures on consumption up to any date τ must be less than total bond holdings plus annuity receipts up to that date, plus expenditures on first period consumption. This constraint can be written as:

$$\sum_{t=1}^{\tau} c_t(1+r)^{1-t} \leq c_1 + \sum_{t=2}^{T} B_t + R_A A \sum_{t=2}^{\tau}(1+r)^{1-t} \quad \forall \tau. \tag{14}$$

This induces one constraint for every period in which consumption is bound from above by the required annuity. Annuities are costly in optimization terms because they contribute to these constraints.

The expenditure minimization problem becomes:

$$\min_{c_1,A,B} c_1 + A + B \tag{15}$$
$$s.t.\ U(c_1, c_2(A,B)) \geq \bar{U}$$
$$s.t.\ \text{equation (14) is satisfied.}$$

Result 5 *If optimal consumption is weakly increasing over time, then complete initial annuitization is optimal. That is, initial net bond purchases are zero.*

Proof. *With non-decreasing consumption, constraint 14 is satisfied when the lifetime budget constraint is satisfied. That is, bonds maturing as needed to satisfy (17) can be purchased from future savings. Hence, if net bond holdings are greater than zero, expenditures can be reduced and utility increased by an additional purchase of ϵ units of A and sale of $\epsilon \frac{R_A}{R_{B2}} > \epsilon$ units of B_2.* ■

Without the annuity, expenditures are given by

$$E(c,0) = c_1 + \sum_{t=2}^{T} c_t R_{Bt}^{-1} = c_1 + \sum_{t=2}^{T} c_t(1+r)^{1-t}. \tag{16}$$

With annuities, the cost of a consumption plan is equal to the cost of annuitized consumption plus the difference between annuitized consumption and actual consumption in every period:

$$E(c,A) = c_1 + A + \sum_{t=1}^{T}(c_t - R_A A)(1+r)^{1-t}, \tag{17}$$

where R_A is the per-period annuity payout. For $t > 1$, if consumption is less than the annuity payout, the difference can be used to purchase consumption at later dates, with the relative prices given by bond returns. If consumption is greater than the annuity payout, then a bond maturing at date t must be purchased.

Adding the assumptions of additively separable preferences over consumption, exponential discounting and access to an actuarially fair constant real annuity generates additional results. If $1 - m_t \equiv \Pi_{s=2}^{t}(1 - q_s)$ is the probability of survival to period t, then actuarial fairness implies that the cost per unit of the annuity is equal to the survival-adjusted present discounted value of bond purchases yielding the same unit per period:

$$1 = \sum_{t=2}^{T}(1 - m_t)\frac{R_A}{(1+r)^{t-1}}$$

$$\Rightarrow R_A = \frac{1}{\sum_{t=2}^{T}(1 - m_t)(1+r)^{1-t}}. \tag{18}$$

Assumption 3 applies as long as there is a positive probability of death by the end of T periods because the cost of consuming any plan $R_A A$ per period past period 1 with annuities is $\frac{A}{\sum_{t=2}^{T}(1-m_t)(1+r)^{1-t}}$. This is less than $\frac{A}{\sum_{t=2}^{T}(1+r)^{1-t}}$, the cost of purchasing A per period with conventional securities.

Here, we assume that utility is given by:

$$U(c_1, c_2) = \sum_{t=1}^{T}\delta^{t-1}(1 - m_t)u(c_t), \tag{19}$$

Where $u' > 0$, $u'' < 0$; $\lim_{c_t \to 0} u' = \infty$, and δ is the rate of time preference.

Result 6 *For the dual utility maximization problem with fixed expenditures, if the optimal level of annuitization A is less than initial wealth savings, so that there are positive initial expenditures on bonds, an increase in δ yields an increase in optimal A relative to savings.*

Proof. *With an increase in δ, for any periods $s' > s$, the ratio of consumption induced by initial period consumption and investment $\frac{c_{s'}}{c_s}$ must increase. This follows since the ratio of marginal utilities increases with δ and the ratio can be increased with a small budget-neutral exchange of B_s for $B_{s'}$. Hence, planned consumption with the increase in δ must be equal to the original consumption plan plus a weakly increasing sequence with negative elements for all dates up to some date t. By the result above, the old consumption plan is revised with minimal expenditures by selling bonds with maturity less than t and increasing A.* ■

Result 7 *If $\delta(1 + r) \geq 1$, complete initial annuitization is optimal.*

Proof. *By result 6, it is sufficient to show that this is true for $\delta(1+r)=1$. For complete annuitization to be suboptimal, it must be the case that there exists some t for which purchasing a bond with maturity at date t provides greater marginal utility than purchase of the real annuity with consumption of each period's annuity receipt, or:*

$$\exists t>1\ :\ \delta^{t-1}(1+r)^{t-1}u'(R_A A)(1-m_t) > \frac{\sum_{t=2}^{T}\delta^{t-1}(1-m_t)u'(R_A A)}{\sum_{t=2}^{T}(1-m_t)(1+r)^{1-t}}.$$

$$\Rightarrow\ \delta^{t-1}(1+r)^{t-1}(1-m_t) > \frac{\sum_{t=2}^{T}\delta^{t-1}(1-m_t)}{\sum_{t=2}^{T}(1-m_t)(1+r)^{1-t}}.$$

If $\delta(1+r)=1$, then this is impossible, because the left hand side is less than or equal to one (by non-negative mortality) and the right hand side equals one. Note that this applies to any later bond purchases so that it is concluded optimal to have constant consumption. ■

If uncertainty were introduced, for complete annuitization to remain optimal, we would require that marginal utility in every state of nature not be so large to justify the cost of adding consumption in that period through a bond rather than adding consumption in every period through the constant real annuity (which we might assume would pay out a constant amount across states of nature as well as periods).

3.2.2 Future Purchase of Annuities and the Possibility of Zero Initial Annuitization

As we have seen, the possibility of future trade in bonds can increase the demand for annuities. Conversely, the possibility of future trades in annuities can decrease the demand for initial annuities, replacing it with a later demand for annuities. Continuing to assume complete bond markets, assume that real annuities can be purchased starting in period one and, in a reopened market, also in period two (this possibility is addressed in Milevsky and Young (2002)). If the internal rate of return (unadjusted for mortality) is larger for the delayed annuity, then it is possible that it is worthwhile to delay annuity purchase, if the survival probability for the first period is large enough.

Consider the case considered above where the only annuity available is a constant real annuity and suppose an individual lives for at most three periods. If the interest rate on bonds is zero, an annuity purchased in period one pays \$0.55 in periods two and three and an annuity purchase in period two pays \$1.50 in period three,[17] then some consumption plans are more cheaply purchased by placing all period one savings in a bond maturing in period two and investing all period two savings in the annuity available in period two.

[17]Such an unrealistic payout scenario could in principle be a product of a selection process whereby early annuitizers are longer lived than late annuitizers.

3.2.3 Incomplete Markets for Nonannuitized Assets and the Possibility of Zero Annuitization

In the original Yaari model, stochastic length of life was the only source of uncertainty. Medical expenses and nursing home costs represent large uncertainties for many consumers. If insurance for these events is incomplete, this will affect the demand for annuities if they are less liquid than bonds or if, for some reason, the available annuities' payouts are relatively large in low marginal utility states. The general incomplete markets sufficient condition guaranteeing positive annuity purchases is that there is an annuity or combination of annuities available which pays out in all the same states of nature as a nonannuitized asset, with payouts that are weakly greater state-by-state. In the real world, this seemingly strong condition could be met by an annuitized version of an underlying asset such as shares in a particular stock or mutual fund. However, with complete Arrow pure bond markets and given survival probabilities, such that price-weighted marginal utility is equated across future states, as long as the optimal plan involves some consumption throughout life and as long as the return condition is satisfied, it remains the case that some annuitization is optimal. Basically, the argument above that the minimal consumption over all possible states and times is best financed by an annuity continues to hold.

With life expectancy as the only risk, individuals can receive information about remaining life expectancy that is not recognized in the market structure. Again, a greater liquidity for bonds would affect annuity demand. In this case, there can be zero demand for annuities if the news implies that the maximal possible length of life has decreased - that is, that the minimal consumption over the initially possible ages is zero. Conversely, if the news changes the probabilities of survival, without shortening the possible maximal life, then some annuitization remains optimal, by the same argument as above. In a three period model with life expectancy news, we derive a necessary condition for zero annuitization.

Suppose that in period 1, a consumer expects to survive to period 2 with probability $1 - q_2$ and to period 3 with probability $(1 - q_2)(1 - q_3)$. However, the consumer knows that in period 2, the conditional probability of survival to period 3 will be updated to zero ("bad health news") with probability α or to $\frac{1-q_3}{1-\alpha}$ ("good health news") with probability $1 - \alpha$. A single compound annuity is available in period one, paying R_{A2} and R_{A3} in periods two and three, respectively. If the bonds fail to distinguish between the two health conditions, the consumer will sell whatever bonds pay off in period three on obtaining bad health news in period two, but will be unable to cash out the illiquid third period annuity claim.

Suppose that without annuitization, the consumer divides period one savings between the bonds maturing in periods two and three such that no trade is undertaken in period two if the consumer obtains good health news. Consumption in period two is thus given by $R_{B2}B_2$ if there is good health news and $R_{B2}B_2 + \frac{R_{B2}}{R_{B3}}R_{B3}B_3$ if the health news is bad. Assume that the consumer's utility is given by $U(c_1, c_2, c_3) = u(c_1) + \delta u(c_2) + \delta^2 u(c_3)$. The marginal utility of savings in either bond is thus:

$$\delta R_{B2}(\alpha u'(R_{B2}(B_2 + B_3)) + (1 - \alpha)u'(R_{B2}B_2). \tag{20}$$

Zero annuity purchase is optimal if and only if expression (20) is greater than or equal to the marginal utility of a small purchase of the annuity. This latter value is simplified by noting that optimal allocation across periods two and three conditional on good health news imply $R_{B2}u'(R_{B2}B_2) = \delta R_{B3}u'(R_{B3}B_3)$. The marginal utility of a small purchase of the annuity is:

$$\delta(\alpha R_{A2}u'(R_{B2}(B_2 + B_3)) + (1 - \alpha)(R_{A2} + R_{A3}\frac{R_{B2}}{R_{B3}\delta})u'(R_{B2}B_2) \tag{21}$$

Expression (20) can exceed expression (21) and hence zero annuitization can be optimal without violating the superior return condition for annuities, here $\frac{R_{A2}}{R_{B2}} + \frac{R_{A3}}{R_{B3}} > 1$. This can occur if the annuities' payouts are sufficiently graded towards future payouts relative to the bonds, the probability of bad health news α is sufficiently large and u is not too concave. Hence, in this particular incomplete markets setting, zero annuitization, partial annuitization and complete annuitization are all consistent with utility maximization without further assumptions.[18].

4 Special Cases: The Welfare Gains from Annuitization with Additive and Standard-of-Living Utility

Much of the literature on annuities has focused on the welfare gains that can be generated by providing access to annuity markets. These simulations have typically assumed that individuals have intertemporally additive utility that exhibits constant relative risk aversion. The gains from annuitization have been shown to be quite substantial. For example, Brown, Mitchell and Poterba (2002) show that a consumer with log utility would find access to an actuarially fair real annuity market equivalent to nearly a 50 percent increase in unannuitized wealth.

[18] In this example zero annuitization cannot be optimal unless the support for being alive changes in period 2. For example, uncertainty about medical expenses might change the extent of annuitization, but would not eliminate annuitization. With psychic or monetary costs to annuitization, demand sufficiently close to zero could result in an optimum no annuitization at all

We saw in Section 3.2.1 that under these conditions, a consumer for whom discounting is a weaker effect than interest rates will annuitize completely. In this section, we discuss the welfare consequences of annuitization in this "industry standard" case. We then expand on the prior literature by examining annuity valuations when utility is no longer intertemporally separable. In particular, we consider a case in which utility is dependent on a standard-of-living, i.e., utility in any period is a function of current and past consumption. We calculate the welfare gains from annuitization under both sets of assumptions and show how a standard-of-living effect can make annuities more or less valuable, depending on how large the initial standard-of-living is relative to available retirement resources. This relationship plays a major role in the level of savings as well as the attractiveness of constant consumption. We consider as in Section 3.2.1 a world with $T-1$ future periods and no uncertainty other than time of death. We evaluate the welfare consequences of the required purchase of A units of an actuarially fair annuity with constant real return R_A in each future period when there are no future opportunities to purchase annuities, but bonds may be purchased both in the present and in the future.

As discussed above, a small increase in A from zero has no effect on consumption, so that the CV from incremental annuitization from 0 to a small number ϵ is equal to the difference between $E(c, 0)$ and $E(c, \epsilon)$:

$$\frac{dE}{dA}|_{A=0} = 1 - \sum_{t=2}^{T}(R_A(1+r)^{1-t}) < 0. \tag{22}$$

The inequality follows from equations (16) and (18) as long as $m_T > 0$.

The welfare effects of larger increases in annuitization are more difficult to sign because they may constrain consumption. Below, we consider the effects for particular utility functions. We also consider the value of a complete annuity market.

4.1 The Gains from Annuitization under "Usual Assumptions"

If utility is additively separable and features exponential discounting, as in specification (19), then the extension to Result 7 follows from the proof above:

Result 8 *If $\delta(1+r) \geq 1$, then any increase in annuitization in the range $A \in [0, E - c_1]$ is welfare enhancing.*

For more impatient consumers (lower δ), we solve for the optimal fraction of savings put into annuities numerically. Results are detailed below.

Beyond the results we have above, making statements about the size of EV for a move from complete annuitization to zero annuitization is difficult, because in general, this calculation must take into account the period-by-period positive wealth constraints summarized in equation (14). That said, a plausible conjecture, based on Result 6 is that valuation will increase in the patience parameter δ, which should push consumption later in life. Further, in cases where optimal consumption is decreasing over time, increased smoothing should increase valuation. Hence, for $\delta(1+r) \leq 1$, we should expect valuation to increase with any parameter of risk aversion, because the desire for decreasing consumption, which makes the liquidity constraints brought on by annuitization bind, would then be tempered by a desire for consumption smoothing. We confirm these intuitions below with numerical examples.

4.2 The Gains from Annuitization when Utility Depends on a standard-of-living

Additive separability of utility does not sit well with intuition. For example, life in a studio apartment with no car is surely more tolerable for someone used to living in a studio apartment without a car than for someone who was forced by a negative income shock to abandon a four bedroom house and an Escalade for a studio apartment and no car. In this section, we consider an extreme and hence illustrative, example of intertemporal dependence in the utility function, taken from Diamond and Mirrlees (2000). The intuition behind this formulation is that it is not the level of present consumption, but the level relative to past consumption that matters. We consider the ratio of present to past consumption, but the difference could also be considered. In choosing how to allocate resources across periods, consumers with such utility trade off immediate gratification from consumption not only against a lifetime budget constraint, but also against the effects of consumption early in life on the standard-of-living later in life.

$$U(c_1, c_2) = \sum_{t=1}^{T} \delta^{t-1}(1-m_t)u(\frac{c_t}{s_t}), \tag{23}$$

where

$$s_t = \frac{s_{t-1} + \alpha c_{t-1}}{1+\alpha}.$$

If individuals' subjective standard of living is constant (i.e. if $\alpha = 0$) we are back in the additively separable case. A positive value of α indicates that past consumption makes individuals less satisfied with a given level of present consumption.

In the absence of the positive wealth constraints (14), the marginal utility of consumption in any period incorporates two effects not present in the additively separable case: (i) the effect of the present standard-of-living on present marginal utility and (ii) the effect of present consumption on future periods' utility through subsequent standards of living. Under this specification, the marginal benefit of present consumption is given by:

$$\frac{\partial U}{\partial c_t} = \delta^{t-1}\frac{1}{s_t}u'(\frac{c_t}{s_t}(1-m_t)) - \sum_{k>t}\delta^{k-1}\frac{\alpha}{(1+\alpha)^{k-t}}\frac{c_k}{s_k^2}u'(\frac{c_k}{s_k})(1-m_k).$$

We note that if $\lim_{\frac{c_t}{s_t}\to 0} u'(c_t) = \infty$, then Assumption 2 holds and Result 4 applies for finite s_1, so some annuitization is optimal.

To do calculations, we assume that $u(\frac{c}{s}) = \frac{(\frac{c}{s})^{1-\gamma}}{1-\gamma}$ and that $\gamma \geq 1$. Hence:

$$\frac{\partial U}{\partial c_t} = \delta^{t-1}c_t^{-\gamma}s_t^{\gamma-1}(1-m_t) - \sum_{k>t}\delta^{k-1}\frac{\alpha}{(1+\alpha)^{k-t}}c_k^{1-\gamma}s_k^{\gamma-2}(1-m_k).$$

For $\gamma > 1$, effect (i) will tend to push consumption towards later periods relative to the no standard ($\alpha = 0$) case if the standard-of-living is increasing over time since a higher standard-of-living increases the marginal utility of consumption. If the standard-of-living is decreasing over time and $\gamma \geq 2$, then effect (i) will tend to push consumption to earlier periods. For $\gamma < 2$, the effect is ambiguous.

Effect (ii) will tend to push consumption towards later periods in life since later consumption raises the standard-of-living in fewer periods. Hence, the result of complete annuitization when the discount rate is less than the interest rate, Result 7, continues to hold if s is constant or decreasing over the period of annuitization. This occurs if the initial value of s is small and the required level of utility, $\bar{U}$, is large. If the initial value s_1 is sufficiently large relative to the expenditures required to attain $\bar{U}$, then the smoothing implied by risk aversion may undo the result by rendering optimal consumption relatively decreasing over time.

With the constraint that the only annuity available pays out a constant real amount, relative valuations are particularly difficult to calculate with standard-of-living effects, because the intertemporal effects compound the difficulty of the multiple positive wealth constraints. However, we conjecture that parameter changes that tend to defer optimal consumption will tend to increase valuation. Hence, simulated valuations should tend to be increasing in δ. Further, large s_1 should yield decreasing valuation and small s_1 increasing valuation, with both effects magnified by γ.

4.3 Numerically Estimated Magnitudes of Welfare Effects

To estimate numerically the value that an individual places on annuitization, we specify that $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ for both the additively separable and standard-of-living effect cases. We assume exponential discounting and a flat yield curve. In the separable case, this gives constant relative risk aversion utility, with a relative risk aversion of γ and an intertemporal rate of substitution of $\frac{\frac{\partial U}{\partial c_t}}{\frac{\partial U}{\partial c_k}} = \delta^{t-k}(\frac{c_t}{c_k})^{-\gamma}\frac{1-m_t}{1-m_k}$. In the standard-of-living effect case, both risk aversion and intertemporal substitution are complicated by the intertemporal utility linkage.

We calculate the utility gains from annuitization for a single, 65 year old male. We use survival probabilities from the U.S. Social Security Administration for the cohort turning age 65 in 1999, modified (to ease computation) so that death occurs for sure by age 100. We use a real interest rate r of 0.03 and vary γ and δ. We normalize wealth at age 65 to be 100 in all cases. We find the consumption vector that solves the expenditure minimization problem numerically using standard optimization techniques.[19]

In Table 1, we report on nine simulations. The first three simulations, in the top panel of the table, report results for a consumer with the usual additively separable utility function. The middle panel contains three simulations for an individual with a standard-of-living utility function. In this case, the consumer retires with a stock of wealth that is 20 times larger than the standard-of-living to which they are accustomed at age 65. Specifically, the consumer has a starting wealth of 100 and standard-of-living s_1 equal to 5. We set α (from equation (23)) equal to 1.

The last three simulations, in the bottom panel, are also for a consumer with preferences that depend on their standard-of-living. In this case, however, the stock of wealth is only twice as large as the standard-of-living to which they are accustomed. Specifically, we set s_1 equal to 50, while we continue to hold wealth at 100 and α equal to 1.

Within each panel, we examine three cases to show how results are affected by γ and δ. The first case in each panel is our "base case" for which $\gamma = 1$ (log utility) and $\delta = 1.03^{-1}$ (and thus is the discount rate is equal to the real interest rate). We then explore how results change when the individual discounts the future more heavily by setting $\delta = 1.10^{-1}$. The third case returns δ to its value of 1.03^{-1} and explores change of γ to a value of 2. Note that for the separable utility cases, γ represents the coefficient of relative risk aversion. While we use the same values of γ for the standard-of-living effect cases, it cannot be interpreted as the risk aversion coefficient. We assume that the consumer cannot borrow against future annuity receipts, but may save annuity payments in bonds with the interest rate of .03.

For each of the nine simulations, we calculate four values. In the first column, we report the equivalent variation (EV) for fully annuitizing in a constant real annuity. In other words,

[19] Inspection of case two shows suboptimally increasing consumption in the last few years of life. The solutions are approximations with only very small deviations from equalized marginal utility to price ratios tolerated for years in which consumption is not equal to the real annuity.

the numbers in column (1) represent the increase in wealth required to hold utility constant while moving all wealth from a constant real annuity to conventional bonds. In the second column, we report the fraction of wealth that is optimally placed in the real annuity instead of bonds if a continuous choice over annuitization levels is permitted. In column (3), we report the equivalent variation associated with the optimal amount of annuitization as reported in column (2). Thus, column (3) represents the increase in wealth required to hold utility constant while moving from having the optimal amount annuitized in a real annuity, to having all of wealth in bonds. The final column reports the gains from annuitization (again in the form of an equivalent variation) for the case in which the individual is permitted to choose an optimal payout trajectory, i.e., they are no longer constrained to purchase a constant real annuity.

In addition to the four welfare measures presented in table 1, we graph the consumption profiles for each of the nine cases in figures 1 through 9. Each graph plots the optimal consumption with different levels and types of annuitization: the series plotted with circles is optimal consumption without annuitization; the series plotted in squares represents optimal consumption with an equivalent utility level, but with 100 percent of wealth (100 units) put into a constant real annuity; the series plotted in triangles represents optimal consumption with the same level of expenditures as in the complete annuitization case (rather than the same level of consumption or utility) but with the consumer free to place an optimal fraction of wealth in the constant real annuity and the remainder in bonds. The series plotted in $\times$s represents optimal consumption when all initial wealth (again 100 units) is placed in annuities which are allowed any desired time shape. A rough estimate of the magnitude of EVs can be obtained by observing the difference in trajectories between the circled consumption plan and the other, annuitized consumption plans. When optimal consumption is sharply decreasing, the constraints implied by (18) bind consumption away from the optimal path. In these cases, the price benefit of annuitization is largely offset by the constraints. When optimal unconstrained (zero annuitization) consumption is hump shaped and less steeply decreasing, the constraints impose less costs, so the net benefit to annuitization is greater.

Turning our attention to the results, we see that the first case is for a consumer with intertemporally separable preferences, log utility and a discount rate equal to the interest rate. For this individual, a constant real annuity provides an optimal consumption path. Therefore, all wealth is annuitized and the EV is the same for columns (1), (3) and (4). Specifically, we find that the individual would require a 44 percent increase in wealth to be made as well off with no annuities as he would be if permitted to use his full wealth to purchase a constant real annuity. This result is very close to those found in the existing literature, despite the truncation of the maximum lifespan at age 100.[20] Figure 1 demonstrates the gains from annuitization graphically, as the consumption path provided by the constant real annuity is optimal given actuarially fair pricing of consumption. Hence, there is no benefit to flexibility in annuity payout trajectory.

The second case considers a different discount factor of 1.10^{-1}, such that the consumer now discounts the future more heavily. This consumer would still prefer to place 100 percent of her wealth in a real annuity than to invest entirely in bonds. However, the gains from full annuitization are much lower, with an EV of only 19. This decline in the value of the annuity arises because the individual would prefer to reallocate consumption from the future to the present, but is essentially liquidity constrained by the constant real nature of the annuity payments, as can be seen in figure 2. Were the individual permitted to annuitize any amount, he would optimally choose to place 72 percent of his assets in the real annuity and retain 28 percent in bonds. If he pursued this strategy, the consumer would have an EV of 19, as indicated in column (3). Column (4) shows the EV for a consumer who is permitted to choose any annuitized payout trajectory that he wishes. We know from Result 1 that complete annuitization is optimal when any consumption stream that can be purchased by bonds can be mimicked by annuities. This number must be weakly greater than the EV associated with complete real annuitization, or equal in the knife-edge case where optimal consumption is constant with actuarially fair prices. In this case, the consumer would choose to place all of his wealth in an annuity with a downward sloping payout trajectory and this would give him an even larger EV of 24.

The final case in the top panel shows the effect of increasing risk aversion from 1 to 2. As has been found elsewhere, this increases the value of annuitization. With a discount factor of 1.03^{-1}, the EV of complete annuitization rises to 56. Complete real annuitization is optimal for this individual.

The three cases in the middle panel consider a standard-of-living effect case, where the individual has a large amount of wealth relative to his standard-of-living. This large ratio of wealth to standard-of-living means that the endowment of wealth is enough to sustain more consumption per year than the consumer is used to. Comparing the results in the middle panel to the upper panel (i.e., no standard-of-living effect), we see that the value of annuitization is much greater. This is not surprising since consumption is backloaded compared with the additively separable cases in the first panel. For the case of log utility and a discount rate equal to the interest rate, EV is 64 for a real annuity and 82 for an

[20]For example, Brown, Mitchell and Poterba (2002) found that the EV for this case was 0.50 when allowing the maximum lifespan to run to age 115.

optimally chosen payout trajectory. Even when the individual discounts the future more highly, annuities are quite valuable, as indicated by the middle case, where the individual would choose to place 99 percent of their wealth in a real annuity. Consistent with the case in which there is no standard-of-living effect, we see that the value of annuitization is increasing with the concavity of the utility function and determined by γ. Figures 4, 5 and 6 graphically show the consumption paths with and without annuitization. The hump shape arises because of the standard-of-living effect. At retirement, the individual has a stock of wealth that allows them to consume in excess of their standard-of-living. Therefore, the individual gradually increases consumption and raises their standard-of-living to a point that it can be sustained given the wealth endowment. The fourth and sixth figures show a considerable difference between optimal consumption with choice over annuity trajectory and given the constant real annuity; hence we see a considerable benefit to flexibility in annuity payout in these cases.

In the bottom panel, we explore the case in which the initial standard-of-living is large relative to resources. In this setting, smoothing the ratio of consumption to the standard-of-living $\frac{c_t}{s_t}$ requires large initial consumption that is rapidly decreasing over time, as indicated in figures 7, 8 and 9. Such a consumption path is inconsistent with a constant real annuity and as a result the standard-of-living effect now reduces the value of the annuity. In the case where γ=1 and $\delta = 1.03^{-1}$, the value of the annuity falls from 44 percent of wealth without the standard-of-living effect to 36 percent with a standard-of-living effect. With a higher discount rate, complete, mandatory real annuitization is even less attractive, providing an EV of only 3. When risk aversion increases to 2, smoothing the ratio $\frac{c_t}{s_t}$ becomes an even greater priority and complete real annuitization actually reduces utility.

Even in the latter case, which is the worst case for annuitization analyzed here, a large fraction of wealth is optimally annuitized even if a constant real annuity is the only form of annuity available. In particular, if the individual is permitted to annuitize 60 percent of wealth in a real annuity, this is equivalent to a 27 percent increase in wealth. For perspective, Mitchell and Moore (2000) find that the median household nearing retirement has pensions and Social Security making up 60 percent of its retirement wealth. Thus, while many households have annuities that make up a substantial fraction of wealth, the implication of these simulations is that preferences alone may have a difficult time explaining the absence of annuitization for households with substantial asset holdings.

5 Conclusions and Future Directions

With complete markets, the result of complete annuitization survives the relaxation of several standard, but restrictive assumptions. Utility need not satisfy the von Neumann-Morgenstern axioms and need not be additively separable. Further, annuities must only offer positive net premia over conventional assets; they need not be actuarially fair. Even with incomplete annuities markets, as long as there is a positive premium to annuitizing wealth and conventional markets are complete, at least some positive fraction of wealth is optimally annuitized.

Even without bequest motives, we find that a lack of complete insurance markets can render even a small amount of annuitization suboptimal. This suggests that an increase in the use of other forms of insurance might encourage annuitization from a demand perspective. This is interesting in light of the suggestion by Warshawsky et al. (forthcoming) that linking annuities and long term care insurance might improve the problem of adverse selection in both markets.

In the much-studied case of a world where only individual mortality is uncertain, we find that there may be considerable individual heterogeneity in the value of annuitization. However, even for preferences which render a constant real annuity relatively unattractive, a large fraction of wealth is optimally annuitized even if this is the only form of annuitization available. It would be interesting to consider for what fraction of the American elderly social security and pensions amount to more than the lowest optimal fraction of wealth (60 %) that we find.

In our simulations, we have retained the abstractions of no bequest motive, no risks other than longevity and no learning about health status or other liquidity concerns. Exploring the consequences of dropping these assumptions in the context of non-separable preferences and unfair annuity pricing would be an important generalization, but obtaining results will require strong assumptions both on annuity returns and on the nature of bequest preferences and liquidity needs.

The near absence of voluntary annuitization and the absence of annuitization early in life are puzzling in the face of theoretical results suggesting large benefits to annuitization. Our analysis extends the puzzle by demonstrating that annuitization of all financial assets is optimal more generally than previously thought. In general, incomplete annuity markets may render annuitization of a large fraction of wealth suboptimal; our simulation results show that this is not the case for some special cases of preferences and when annuity markets are incomplete only in that they impose a single payout trajectory across time.

It is sometimes argued that the lack of annuity purchase is evidence for bequests. This raises the question of what sort of bequest motive would call for an absence of annuities. If there is no annuitization, then a bequest is random in both timing and size, measured as a PDV. Assuming one cares about the risk aversion of recipients, this may be dominated by giving the heirs a fixed sum at a fixed time and annuitizing the rest. More generally, partial annuitization can reduce the variation in the bequest.[21] The extent of dominance depends on load factors; with a bequest motive, the load factor that is sufficient to cut off annuity purchases is lower, because we expect that sharing the outcome with someone else reduces risk aversion.

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[21]This is similar to the case made against use of a years certain annuity.

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Table 1: Simulated Utility Gains from Access to Annuitization
(EV = Equivalent Variation)

Case	(1) EV for all wealth in a constant real annuity	(2) Optimal % of wealth annuitized in real annuity	(3) EV for optimal % of wealth in a real annuity	(4) EV for all wealth in an annuity with optimal payout trajectory
No Standard of Living				
$\gamma = 1$				
$\delta = 1.03^{-1}$	44	100%	44	44
$\delta = 1.10^{-1}$	15	72%	19	24
$\gamma = 2$				
$\delta = 1.03^{-1}$	56	100%	56	56
"Small" Standard of Living (ratio of wealth to standard of living = 20)				
$\gamma = 1$				
$\delta = 1.03^{-1}$	64	100%	64	83
$\delta = 1.10^{-1}$	36	99%	36	38
$\gamma = 2$				
$\delta = 1.03^{-1}$	70	100%	70	91
"Large" Standard of Living (ratio of wealth to standard of living = 2)				
$\gamma = 1$				
$\delta = 1.03^{-1}$	36	84%	45	48
$\delta = 1.10^{-1}$	3	63%	20	24
$\gamma = 2$				
$\delta = 1.03^{-1}$	-9	60%	35	41

Notes: As explained in text, all simulations are for a 65-year old male with a starting wealth normalized to 100. Mortality rates are based on cohort tables the cohort table for men turning 65 in 1999, as determined by the Office of the Actuary of the Social Security Administration. Column 1 reports the equivalent variation for placing 100 percent of wealth in a constant real annuity. Column 2 reports the optimal fraction of wealth that would be annuitized in a real annuity if the individual is given the choice to partially annuitize. Column 3 reports the equivalent variation for placing in a real annuity the fraction of wealth reported in column 2. Column 4 reports the equivalent variation for complete annuitization when the individual can choose any payout trajectory. In the case of no standard of living, γ represents the coefficient of relative risk aversion. δ is the discount factor.

Figure 1: Optimal Consumption by age past retirement for different levels of annuitization: $U = \sum_{t=1}^{35} 1.03^{-t} ln(c_t)(1 - m_t)$

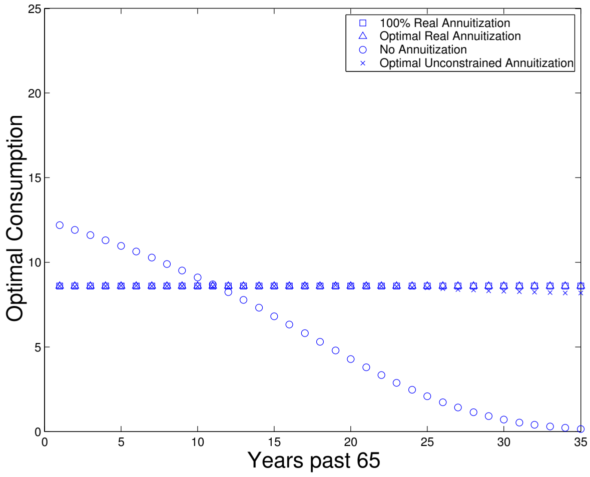

Figure 2: Optimal Consumption by age past retirement for different levels of annuitization: $U = \sum_{t=1}^{35} 1.1^{-t} ln(c_t)(1 - m_t)$

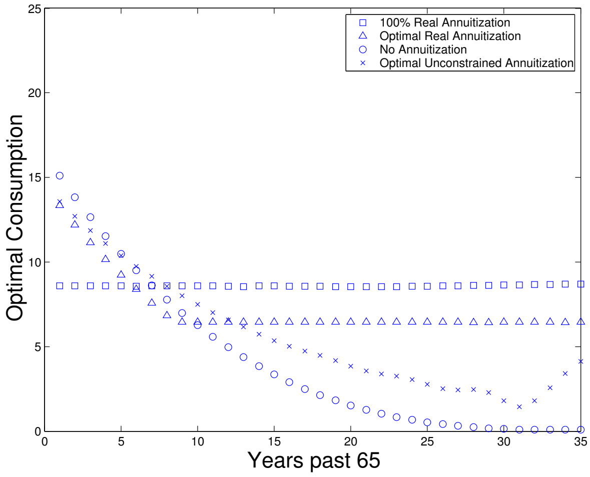

Figure 3: Optimal Consumption by age past retirement for different levels of annuitization: $U = \sum_{t=1}^{35} -1.03^{-t} c_t^{-1}(1 - m_t)$

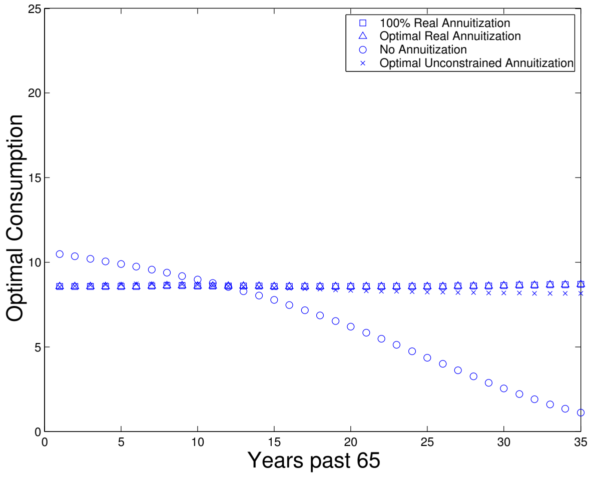

Figure 4: Optimal Consumption by age past retirement for different levels of annuitization: $U = \sum_{t=1}^{35} 1.03^{-t} ln(\frac{c_t}{s_t})(1 - m_t),\ s_0 = 5$

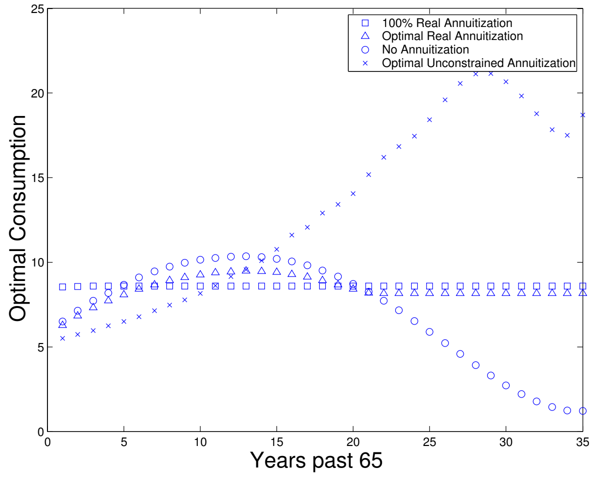

Figure 5: Optimal Consumption by age past retirement for different levels of annuitization: $U = \sum_{t=1}^{35} 1.1^{-t} ln(\frac{c_t}{s_t})(1 - m_t),\ s_0 = 5$

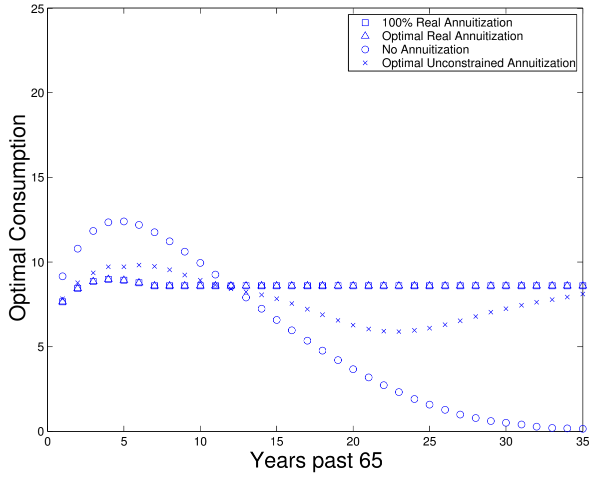

Figure 6: Optimal Consumption by age past retirement for different levels of annuitization: $U = \sum_{t=1}^{35} -1.03^{-t} (\frac{c_t}{s_t})^{-1}(1 - m_t),\ s_0 = 5$

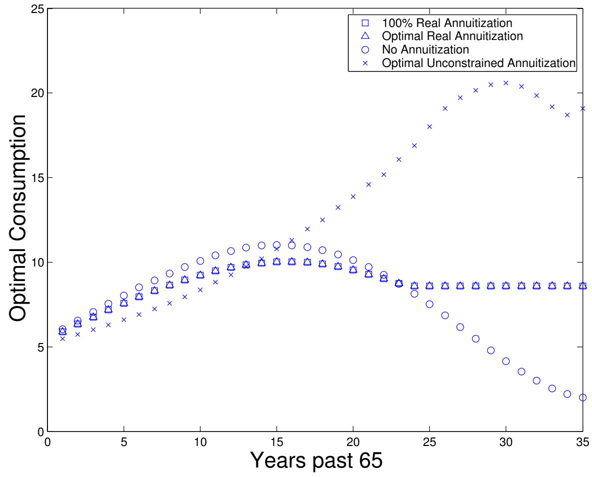

Figure 7: Optimal Consumption by age past retirement for different levels of annuitization: $U = \sum_{t=1}^{35} 1.03^{-t} ln(\frac{c_t}{s_t})(1 - m_t),\ s_0 = 50$

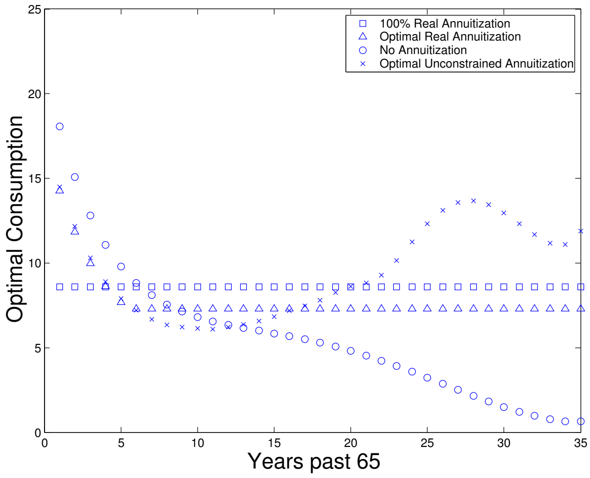

Figure 8: Optimal Consumption by age past retirement for different levels of annuitization: $U = \sum_{t=1}^{35} 1.1^{-t} ln(\frac{c_t}{s_t})(1 - m_t),\ s_0 = 50$

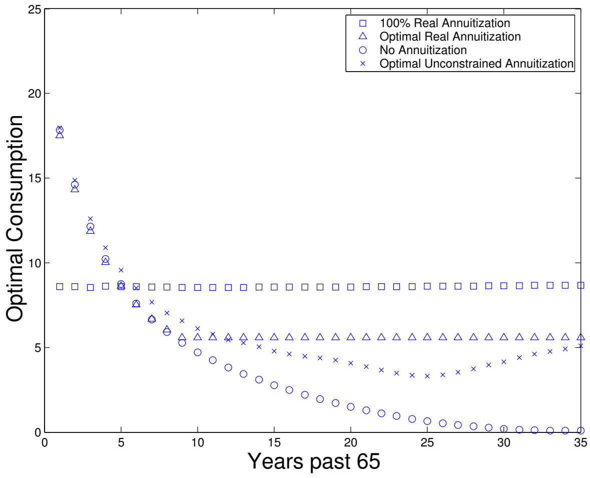

Figure 9: Optimal Consumption by age past retirement for different levels of annuitization: $U = \sum_{t=1}^{35} -1.03^{-t}(\frac{c_t}{s_t})^{-1}(1 - m_t),\ s_0 = 50$